PROJECTED WRITTEN NOTES FROM THE M 408 LECTURE OF TUESDAY, FEBRUARY 20, 2024, on

MORE ON POWER SERIES REPRESENTATIONS (PSRs)

CLASS # 11

DISCOVERING A PSR for a given function f(x).

PROBLEM: (Showing Method ② from the Handout)

Determine a PSR for $f(x) = \frac{1}{3+x^2}$.

Solution: $\frac{1}{3+x^2} = \frac{1}{3(1+\frac{x^2}{3})} = \frac{1}{3(1-(-\frac{x^2}{3}))}$

$$f(x) = \frac{1}{3+x^2} = \frac{1}{3}\left(\frac{1}{1-(-\frac{x^2}{3})}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n \qquad \boxed{r = -\frac{x^2}{3}}$$

$$\frac{1}{3+x^2} = \frac{1}{3}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^n}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^{n+1}}$$

$$f(x) = \frac{1}{3+x^2} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^{n+1}} \text{ with Radius } R = \underline{\sqrt{3}}$$

Determing R, $R = ?$

$|r| < 1$. Here, $\left|-\frac{x^2}{3}\right| < 1 \Rightarrow \left|\frac{x^2}{3}\right| < 1$

$\Rightarrow \frac{|x|^2}{3} < 1 \Rightarrow |x|^2 < 3 \Rightarrow \underline{|x| < \sqrt{3}}$

$|x-0| < \sqrt{3}$, $\quad |x-a| < b \Rightarrow R = b$.

Problem: (Showing method ①)

Use the PSR from the previous example to find a PSR for $g(x) = \frac{x^6}{3+x^2} = x^6 \cdot f(x)$

Sol'n: $\frac{x^6}{3+x^2} = x^6\left(\frac{1}{3+x^2}\right) = x^6\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{3^{n+1}}\right)$

$$g(x) = \frac{x^6}{3+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+6}}{3^{n+1}} \quad \text{with } R = \sqrt{3}$$

When you multiply a power series $\sum_{n=0}^{\infty} c_n x^n$ that has Radius R by x^k, then the new powerseries

$\sum_{n=0}^{\infty} c_n x^{(n+k)}$ has the same Radius of convergence R.

THEOREM 2 on p. 789

When a differentiable function $f(x)$ is expressed as a power series with a particular Interval of Convergence with RADIUS OF CONVERGENCE R,

THEN

Its Derivative $f'(x)$ and its integral $\int f(x)dx$ can also be expressed by power series.

THIS REPRESENTATION SERIES WILL HAVE THE SAME RADIUS R AS RADIUS OF CONVERGENCE (BUT THE INTERVAL OF CONVERGENCE MAY DIFFER AT THE ENDPOINTS)

AND THE NEW SERIES ARE OBTAINED FROM THE ORIGINAL BY USING "TERM-BY-TERM" DIFFERENTIATION OR INTEGRATION.

$\therefore$ When $f(x) = \sum_{n=0}^{\infty} c_n x^n$,

$$f'(x) = \sum_{n=0}^{\infty} n \cdot c_n x^{n-1}$$ and

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} + C$$

SCHEMATIC SHOWING METHODS (3) and (4) of

PSR DISCOVERY

Given $f(x)$, FIND A PSR $\sum_{n=0}^{\infty} c_n x^n = f(x)$ ← OR SOME OTHER POSITIVE INTEGER

METHOD (3)

START HERE: LOOK AT $f(x)$. $f(x) = G'(x)$ where $G(x)$ has a PSR

$$\int (-)\,dx$$

$\int f(x)\,dx = G(x) + C$. FIND PSR, $G(x) = \sum_{n=0}^{\infty} c_n x^n$ with RADIUS R

Differentiate TERM BY TERM

$$\frac{d}{dx}(-)$$

$f(x) = \sum_{n=0}^{\infty} c_n (n x^{n-1})$ with Radius R

Simplify, if needed.

Method (4).

START HERE. Look at $f(x)$

$$\frac{d}{dx}(-)$$

$f'(x)$. FIND a PSR $f'(x) = \sum_{n=0}^{\infty} c_n x^n$ with Radius R

Integrate Term by Term

$$\int (-)\,dx$$

$f(x) = \sum_{n=0}^{\infty} c_n \frac{1}{n+1} x^{n+1} + C$ with Radius R.

Solve for the Correct Value of C.

A Method ④ EXAMPLE (FOR when $f'(x)$ has a PSR)

Problem: Find the PSR for $f(x) = \ln(1+x)$.

$\swarrow (-x)^n \quad r = -x \quad |-x| < 1 \quad |x| < 1$

Sol'n:

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{with } R = 1$$

$$f(x) = \ln(1+x) = \int f'(x)\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}\right) + C$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} + C$$ for the Right C with Radius $R = 1$.

Getting C: At $x = 0$ (your choice), $f(0) = \ln(1+0) = \ln(1) = 0$

and $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} + C = 0 + C = C$ $\nwarrow x = 0$

$\Rightarrow 0 = C$

$C = 0$ is the correct C.

A PSR: $\displaystyle \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}$, $R = 1$ $\quad \rightarrow \displaystyle\sum_{\substack{n-1=0\\ n=1}}^{\infty}$

Goal: Replace $n+1$ with n.

$-1 \qquad -1$

TASK: Replace n with $n-1$ Everywhere

$$\boxed{\text{PSR: } \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n} x^n, \quad R = 1}$$

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} x^{6n+1} \right) \Bigg]_0^{0.3}$$

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} (0.3)^{6n+1} = \underset{S}{\text{Summation}}$$

We know this series converges since $-1 < 0.3 < 1$.

This series is an alternating series

$S = b_0 - b_1 + b_2 - b_3 + \cdots \qquad b_n \, |a_n|$.

$$S = 0.3 - \frac{1}{7}(0.3)^7 + \frac{1}{13} x^{13} - \frac{1}{19} x^{19} + \cdots$$

For Alternating series, the |error| in $S_n \approx S$
is $b_{n+1} = |a_{n+1}|$.

$$b_1 = \frac{1}{7}(0.3)^7 = 0.00003124 3$$

$$b_2 = \frac{1}{13}(0.3)^{13} = 0.\underline{000}\,\underline{000}\,012 < 0.\underline{000000}5$$

Use $S_1 \approx S$, $S_1 = b_0 - b_1 = 0.3 - \frac{1}{7}(0.3)^7$
$\approx 0.2999687\ldots$

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx = 0.299969 \text{ correct to 6 decimal places.}$$

Problem: Find an Approximation of

$$\int_0^{0.3} \frac{1}{1+x^6}\,dx$$ that is correct to 6 decimal places.

Sol'n: We need to have $|ERROR| < 0.0000005$.

Express the Integrand as a power series.

Let $$f(x) = \frac{1}{1+x^6} = \frac{1}{1-(-x^6)} = \sum_{n=0}^{\infty} (-x^6)^n$$

$r = -x^6$

$|-x|^6 < 1$

$|x|^6 < 1$

$|x| < \sqrt[6]{1}$

$|x| < 1$

$R = 1$

$$f(x) = \frac{1}{1+x^6} = \sum_{n=0}^{\infty} (-1)^n x^{6n}, \quad R = 1$$

$$\int \frac{1}{1+x^6}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{6n} \right) dx$$

$$\int \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+1} x^{6n+1} + C,$$

$R = 1$

Last time, we saw that

a PSR for $f(x) = \dfrac{1}{(7-x)^2} = \sum_{n=0}^{\infty} \dfrac{1}{7^{n+2}} (n+1) x^n$

$R = 7, \quad |x| < 7$

<u>Task:</u> Find a PSR for $\dfrac{1}{(7-x^3)^2}$.

Here, in $f(x)$, we replaced x with x^3,
and we do it in the summation too.

NOTE: Replacing x with x^3 changes the Radius R value.

$$f(x^3) = \frac{1}{(7-x^3)^2} = \sum_{n=0}^{\infty} \frac{1}{7^{n+2}} (n+1) (x^3)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{7^{n+2}} (n+1) x^{3n}, \quad \underline{R = \sqrt[3]{7}}$$

FINDING R

$$\left\{ \begin{array}{l} |x^3| < 7 \Rightarrow |x|^3 < 7 \\ |x| < \sqrt[3]{7} \Rightarrow R = \sqrt[3]{7} \end{array} \right.$$

NOTATION CONVENTION: Given function $f(x)$,

$f'(x)$ = first derivative

$f^{(1)}(x) = f'(x)$

$f^{(2)}(x) = f''(x)$ also $f^{(0)}(x) = f(x)$

$f^{(3)}(x) = f'''(x)$

$\vdots$

$f^{(n)}(x)$ = the nth derivative.